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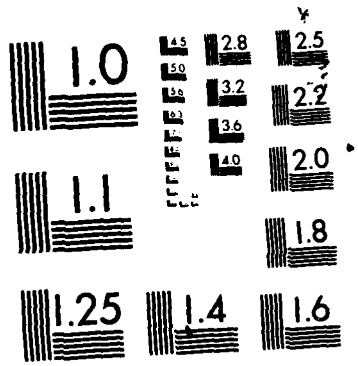

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A GLOBAL BRANCH OF STEADY VORTEX RINGS

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MATHEMATICS RESEARCH CENTER

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ABSTRACT

A steady vortex ring of prescribed strength and propagation speed can be described in terms of a Stokes stream function Ψ. A flux constant k measures the flow through the center of the axisymmetric vortex ring. For $k = 0$, Hill in 1894 found an explicit solution for the semi-linear elliptic equation satisfied by Ψ. In this paper it is shown that there is an unbounded, closed, connected branch of solutions emanating from Hill's vortex in the space of pairs (k,Ψ).

AMS (MOS) Subject Classifications: 76C05, 35B32, 35R05, 35R35.

Key Words: Steady vortex rings, degree theory, global bifurcation.

Work Unit Number 1 - Applied Analysis.

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SIGNIFICANCE AND EXPLANATION

A number of existence theorems for steady vortex rings and some properties of solutions have been established in the last fifteen years. However it is not known whether the vortex rings found for various physical parameter ranges can be connected through parameter changes. Numerical calculations indicate that the known vortex rings are so connected. In this paper it is established that there is an unbounded, connected branch of vortex rings emanating from the well-known Hill's vortex. This supports the results of numerical calculations and paves the way toward establishing specific characteristics along the branch.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

A GLOBAL BRANCH OF STEADY VORTEX RINGS

C. J. Amick[1] and R. E. L. Turner[2]

1. Introduction

The physical problem under consideration in this paper concerns steady vortex rings in an ideal fluid occupying all of $\mathbb{R}^3$. A more complete description of the physical problem may be found in [11] and other existence results in [2], [6] - [10], [12], [19] - [21]. The only explicit solution known is that due to Hill [15] in 1894 and our purpose is to prove the existence of an unbounded, closed, connected branch of solutions emanating from it.

An axisymmetric flow is sought and thus the independent coordinates are taken to lie *in the half plane*

$$\Pi = \{(r,z) : r > 0, -\infty < z < \infty\} \quad .$$

The mathematical problem is to find a flux parameter $k > 0$; a bounded, open vortex "core" $A \subset \Pi$; and a stream function $\Psi = \Psi(r,z) \in C^1(\overline{\Pi}) \cap C^2(\Pi - \partial A)$ such that

$$L\Psi \equiv r\left(\frac{1}{r}\Psi_r\right)_r + \Psi_{zz} = \begin{cases} -\lambda r^2 & \text{in } A \ , \\ 0 & \text{in } \Pi - \overline{A} \ , \end{cases} \tag{1.1}$$

$$\Psi|_{\partial A} = 0 \quad , \quad \Psi|_{r=0} = -k \tag{1.2}$$

and

$$\Psi(r,z) + \frac{1}{2}Wr^2 + k \to 0, \ \frac{\Psi_z}{r} \to 0, \ \frac{\Psi_r}{r} \to -W \tag{1.3}$$

as $r^2 + z^2 \to \infty$ in $\overline{\Pi}$. The vortex-strength parameter $\lambda > 0$ and the propagation speed $W > 0$ are given.

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The use of the strong maximum principle in conjunction with (1.1) - (1.3) shows that $\Psi < 0$ in $\Pi - \bar{A}$ and $\Psi > 0$ in A so that the core is

$$A = \{(r,z) \in \Pi : \Psi(r,z) > 0\} \quad .$$

In cylindrical coordinates (r,θ,z), the velocity field $\vec{q}$ has the components $-\psi_z/r$, 0, ψ_r/r, respectively. Since $\vec{q}$ is to be continuous, the condition $\Psi \in C^1(\bar{\Pi})$ is to be expected. The vorticity, curl $\vec{q}$, has cylindrical components $(0, -(L\Psi)/r, 0)$, so that (1.1) gives a jump in vorticity across ∂A. This causes a jump in the second derivatives of Ψ across ∂A and allows one to have Ψ smooth merely in $\Pi - \partial A$.

For any value of k, the function $\Psi = -\frac{1}{2} Wr^2 - k$ satisfies (1.1) - (1.3) with $A = \phi$; these are trivial solutions, and we shall be interested in non-trivial solutions. With the aid of the Heaviside function

$$f_0(t) = \begin{cases} 0 \quad , & t < 0 \\ 1 \quad , & t > 0 \end{cases} \tag{1.4}$$

the equation (1.1) may be expressed in the form $L\Psi = -\lambda r^2 f_0(\Psi)$. Define ψ by the formula

$$\Psi(r,z) = \psi(r,z) - \frac{1}{2} Wr^2 - k \quad .$$

Then the equation (1.1) becomes

$$L\psi = -\lambda r^2 f_0(\psi - \frac{1}{2} Wr^2 - k) \quad \text{in} \quad \Pi \tag{1.5}$$

and the boundary conditions can be expressed roughly as

$$\psi \to 0 \quad \text{on} \quad \partial\Pi \tag{1.6}$$

by which one should understand that $\psi = 0$ for $r = 0$ and ψ and $\frac{1}{r} \nabla\psi$ approach zero at infinity. These conditions are not made more precise here for the problem will be reformulated in the next section in such a way as to prescribe a precise function space for ψ. In (1.5), (1.6) the numbers λ and W are still prescribed and a solution pair (k,ψ), with $k \geq 0$, is to be determined. Implicit in (1.5), (1.6) is that

$$A = \{(r,z) : \psi(r,z) > \frac{1}{2} Wr^2 + k\}$$

is to be a bounded, open subset of Π. The problem (1.5), (1.6) is a special case of a more general problem in which one replaces f_0 by a general vorticity-distribution

function f. A class of distributions, other than the Heaviside function, which has received considerable attention in recent years consists of functions which are zero for non-positive arguments, smooth, and non-decreasing. The papers [2], [7], [11], [12] and [19] include results for this class, proved largely by variational techniques.

For the Heaviside vorticity distribution and flux $k = 0$ Hill [15], nearly a century ago, found the solution

$$\psi_H(r,z) = \begin{cases} \frac{1}{2} Wr^2\left(\frac{5}{2} - \frac{3}{2}\frac{\rho^2}{a^2}\right), & 0 < \rho < a \ , \\ \frac{1}{2} Wr^2\, a^3/\rho^3 & , \ \rho > a \ , \end{cases} \tag{1.7}$$

where $\rho^2 = r^2 + z^2$ and

$$\lambda a^2 = 15W/2 \ . \tag{1.8}$$

The core A for Hill's solution is merely a semi-circle of radius a outside of which the vorticity vanishes. In [3] it was shown that ψ_H is the unique solution when viewed in a natural weak formulation (cf (1.10)).

For small $k > 0$ Norbury [20] proved that there are solutions near Hill's vortex, each solution having a core homeomorphic to a torus when viewed in R^3. The analysis in [20] was based on a contraction principle in a ball of radius k centered at Hill's solution and hence other solutions near $(0,\psi_H)$ were not ruled out. In [4] it was shown that this local branch of solutions (k,ψ) emanating from $(0,\psi_H)$ constitute the only solutions in a neighborhood of $(0,\psi_H)$ for prescribed positive values of λ and W. Consequently, the local branch will be a subset of the global branch we find in this paper. A global branch is suggested by the numerical calculations of Norbury [21].

A result of Esteban [8] for solutions of (1.5), (1.6) with quite general, but smooth, vorticity distributions f, shows that solutions must always be symmetric about a line z = constant. An analogous result holds for the Heaviside distribution f_0: one uses the arguments of Gidas, Ni, and Nirenberg [13] with the extension given in [3]. Hence, without

loss of generality, we may assume our solutions are even functions of z. An inner product occurring naturally in conjunction with the operator L is

$$\langle\phi,\psi\rangle_H = \int_\Pi \frac{1}{r^2} [\phi_r\psi_r + \phi_z\psi_z) r\,dr\,dz \quad . \tag{1.9}$$

The space in which solutions are sought is $H(\Pi)$, the completion of the functions in $C_0^\infty(\Pi)$, even in z, in the norm corresponding to the inner product (1.9). With this notation, equations (1.5), (1.6) have the weak formulation

$$\langle\phi,\psi\rangle_H = \lambda \int_\Pi f_0(\psi - \frac{1}{2} Wr^2 - k)\phi \; r\,dr\,dz \quad , \quad \forall\, \phi \in H(\Pi) \tag{1.10}$$

and a solution (k,ψ) is to be understood in the sense of (1.10).

The main result of the paper can now be stated.

Theorem 1.1. Let $\lambda > 0$ and $W > 0$ be given and let ψ_H be Hill's solution (1.7), (1.8).

(a) There exists an unbounded, closed, connected set $C \subset [0,\infty) \times H(\Pi)$ of solutions (k,ψ) of (1.10) with $C \cap (\{0\} \times H(\Pi)) = \{(0,\psi_H)\}$.

(b) There exists an $\varepsilon > 0$ and a continuous function $g : [0,\varepsilon] \to H(\Pi)$ with $g(0) = \psi_H$ such that $\{(k,g(k)) : k \in [0,\varepsilon]\} \subset C$ and constitutes the only solutions of (1.10) in a neighborhood of $(0,\psi_H)$.

(c) If $(k,\psi) \in C$, then the following hold: The vortex core $A = \{(r,z) : \psi(r,z) > \frac{1}{2} Wr^2 + k\}$ is bounded; $\psi \in C^{1+\alpha}(\bar{\Pi}) \cap C^2(\Pi - \partial A)$ for any $\alpha \in (0,1)$; ψ is an even function of z; $\psi_z(r,z) < 0$ for $z > 0$; and at infinity, $\psi = O(1/\sqrt{r^2 + z^2})$ and $|\nabla\psi| = O(1/(r^2 + z^2))$.

The new aspect of the theorem is part (a). Part (b) is the main result of [4] while (c) is standard from the estimates of Fraenkel and Berger [11]. To prove (a), we begin section 2 with a further reformulation of the problem. The change of variables $\psi(r,z) = r^2 v(r,z)$ is made, and if v is considered as a function in $\mathbb{R}^5$ with $r^2 = \sum_{i=1}^{4} x_i^2$ and $z = x_5$, then the operator corresponding to L is the Laplacian. This fortuitous fact has been used in [3] and [19] in analyzing the vortex ring problem. If the Laplacian is

formally inverted, a functional equation of the type $v - N(k,v) = 0$ arises. Degree theory and global bifurcation methods suggest themselves. However, two difficulties are encountered. First, the underlying domain is the whole space R^5 and so the inverse of the Laplacian is not compact This is handled by working first in a ball of radius b in R^5. Second, the intervening function f_0 is discontinuous, making N discontinuous. By approximating f_0 by a continuous function f_δ which converges to f_0 as $\delta \to 0$, a continuous and differentiable map is obtained. In this setting, for $k = 0$, a degree computation is made about a solution $v_{b,\delta}$ of the altered problem. The degree is shown to be -1 from which one concludes that a branch of solutions emanates from $(0,v_{b,\delta})$. In section 3 we return to the original problem by letting $\delta \to 0$ and then letting $b \to \infty$ showing in the process that the desired continuum of solutions results. In section 4 we consider the nature in which this set of solutions is unbounded. The numerical calculations of Norbury [21] suggest that the branch extends to $k = \infty$ and that solutions with large k approach a class of solutions examined by Fraenkel [9], [10]. We conjecture that the branch is unbounded in the k-direction and provide some evidence to show that if this is false, then the solutions converge to a 'singular' solution which is a function of r alone and has infinite norm.

2. An Equivalent Problem

2.1. Preliminaries

The purpose of this section is to derive a transformed problem in $\mathbb{R}^5$ which is equivalent to (1.10). The notation follows that from [3]. Let $\lambda > 0$, $W > 0$ be fixed and let $(k,\psi) \in [0,\infty) \times (H(\Pi) \cap C^2(\Pi-\partial A))$ satisfy (1.5). We first rescale variables by setting $\tilde{k} = 2k/Wa^2$, with a as in (1.8), and $\tilde{\psi}(r,z) = 2\psi(ar,az)/a^2W$ so that $(\tilde{k},\tilde{\psi})$ satisfies

$$L\psi = -15r^2 f_0(\psi - r^2 - k) \quad . \tag{2.1}$$

In the sequel we restrict attention to (2.1) and its corresponding weak form.

Consider $\mathbb{R}^5$ and let (r,z) with $r^2 = \sum_{i=1}^{4} x_i^2$ and $z = x_5$ be cylindrical coordinates. Under the change of variables $\psi(r,z) = r^2 v(r,z)$, there results

$$\frac{1}{r^2} L\psi = \Delta v \tag{2.2}$$

where Δ, throughout the paper, denotes the Laplacian in $\mathbb{R}^5$. Hence (2.1) corresponds to

$$\Delta v = -15 f_0(v - 1 - \frac{k}{r^2}) \quad \text{in} \quad \mathbb{R}^5 \quad . \tag{2.3}$$

The occurrence of the Laplacian was used in a crucial way in [3] and [19], and will be the key to a tractable analysis here. We anticipate bounded solutions of (2.3) and thus $\psi = r^2 v$ will vanish at $r = 0$. The condition

$$|v| \to 0 \quad \text{as} \quad |x| \to \infty \quad \text{in} \quad \mathbb{R}^5 \tag{2.4}$$

is allied to (1.6). Indeed, the function space setting for v will ultimately give decay of the order $|x|^{-3}$ so that $\psi = r^2 v$ approaches zero at infinity.

Note that from the form (2.3) with $k = 0$ it is a simple matter to write down Hill's solution. In this setting it is the radial function

$$v_H(r,z) = \begin{cases} \frac{5}{2} - \frac{3}{2}\rho^2 & 0 < \rho < 1 \ , \\ 1/\rho^3 & \rho > 1 \ , \end{cases} \tag{2.5}$$

where $\rho^2 = r^2 + z^2 = \sum_{i=1}^{5} x_i^2$. When $k = 0$ the analysis in [3] shows that the methods of

Gidas, Ni and Nirenberg [13] are applicable and so any solution is a radial function. Thus Hill's solution in (2.5) is unique.

Let ψ and ϕ be in functions in $C_0^\infty(\Pi)$ and let v and u be cylindrically symmetric functions on $\mathbb{R}^5$ (that is, depending only on r and z) defined by $\psi(r,z) = r^2 v(r,z)$ and $\phi(r,z) = r^2 u(r,z)$, as earlier in this section. Let

$$\langle u,v\rangle_E = \frac{1}{2\pi^2}\int_{\mathbb{R}^5} \nabla u(x)\cdot\nabla v(x)dx \quad . \tag{2.6}$$

Then from (2.2) or from a direct computation one has

$$\langle \phi,\psi\rangle_H = \langle u,v\rangle_E \quad .$$

It is natural to define a space E_c as the completion with respect to the norm $\|\ \|$ induced by $\langle\cdot,\cdot\rangle_E$ of the cylindrically symmetric functions in $C_0^\infty(\mathbb{R}^5)$ which are even in z. The following lemma follows from section 2.2 of [3].

Lemma 2.1. (a) The spaces $H(\Pi)$ and E_c are isometrically isomorphic under the correspondence $\psi \leftrightarrow r^2 v$.

(b) A pair $(k,\psi) \in [0,\infty) \times H(\pi)$ satisfies the weak equation

$$\langle \phi,\psi\rangle_H = 15\int_\Pi f_0(\psi - r^2 - k)\phi r dr dz \tag{2.7}$$

for all $\phi \in H(\Pi)$ if and only if $(k, v = \frac{\psi}{r^2}) \in [0,\infty) \times E_c$ satisfies

$$\langle u,v\rangle_E = \frac{15}{2\pi^2}\int_{\mathbb{R}^5} f_0(v - 1 - \frac{k}{r^2})u dx \tag{2.8}$$

for all $u \in E$.

Lemma 2.1(b) shows that it will suffice for Theorem 1.1 to show there exists an unbounded, closed, connected set D of solutions of (2.8) with $D \cap (\{0\} \times E_c) = \{(0,v_H)\}$.

2.2. The transformed problem on bounded domains

Let $B(b) = \{x \in \mathbb{R}^5 : |x| < b\}$ and in analogy with the definition of E_c, let $E(b)$ ($E_c(b)$) be the completion in the E norm of the (cylindrically symmetric) functions in $C_0^\infty(B(b))$ which are even in z. Corresponding to (2.8) is the problem: Find $v \in E_c(b)$ such that

$$\langle u,v\rangle_E = \frac{15}{2\pi^2}\int_{B(b)} f_0\left(v - 1 - \frac{k}{r^2}\right)u, \quad \forall\, u \in E(b) \quad . \tag{2.9}$$

Its solution can be regarded as a fixed point problem. For given $k > 0$ and $v \in E_c(b)$ let $w \in E(b)$ be the solution of

$$\langle u,w\rangle_E = \frac{15}{2\pi^2}\int_{B(b)} f_0\left(v - 1 - \frac{k}{r^2}\right)u, \quad \forall\, u \in E(b) \quad . \tag{2.10}$$

The existence and uniqueness of w follows from standard elliptic theory [14]. That $w \in E_c(b)$ follows from the arguments for Lemma 2.3(b) in [3]. If w is denoted by $N(k,v;b)$ then a fixed point of N is a solution of (2.9). The function w is a weak solution of $-\Delta w = 15f_0(v - 1 - k/r^2)$ in $B(b)$ and since $|\Delta v| \leq 15$ standard regularity theory [1] gives $w \in W^{2,p}(B(b)) \cap \overset{\circ}{W}{}^{1,2}(B(b))$ for all $p \in [1,\infty)$. Sobolev embeddings then yield $w \in C^{1+\alpha}(\overline{B(b)})$ for all $\alpha \in (0,1)$. If $A = \{x \in R^5 : v > 1 + k/r^2\}$, then with $u = w$ in (2.10) one obtains

$$\begin{aligned} \|w\|^2 &\leq \frac{15}{2\pi^2}\int_A |w| \\ &\leq \frac{15}{2\pi^2}\left(\int_A |w|^{10/3}\right)^{3/10} \cdot |A|_5^{\,7/10} \\ &\leq \text{const.}\ \|w\| \cdot b^{7/2} \end{aligned} \tag{2.11}$$

where $|\ |_5$ denotes Lebesque measure in R^5 and the constant is independent of b, w, and k (see Lemma 2.1 of [3]). Since $\|w\| \leq \text{const}\ b^{7/2}$, the elliptic regularity arguments now insure that the bounds on w in $W^{2,p}(B(b))$ and in $C^{1+\alpha}(B(b))$, which depend on b, p, and α, will, however, be independent of k and v.

The solutions of $v - N(0,v;b) = 0$ are known explicitly from [3]. One is

$$v_b(r,z) = \begin{cases} \dfrac{1}{1-c}\left(\dfrac{5}{2} - c - \dfrac{3}{2}\dfrac{\rho^2}{a^2}\right), & 0 \leq \rho \leq a \quad , \\[2ex] \dfrac{1}{1-c}\left(\dfrac{a^3}{\rho^3} - c\right), & a \leq \rho \leq b \quad , \end{cases} \tag{2.12}$$

where $a = 1 + \dfrac{1}{b^3} + O(b^{-6})$ is the smaller root of

$$a^2\left(1 - \frac{a^3}{b^3}\right) = 1 \tag{2.13}$$

and $c = a^3/b^3$. We assume $b > \left(\frac{5}{3}\right)^{1/2} \left(\frac{5}{2}\right)^{1/3}$ so that (2.13) has two distinct roots $a_1(b) < a_2(b)$ in $(0,b)$. Corresponding to the root a_2 is a solution $\tilde{v}_b$ with $a = a_2$ in (2.12). Note that while $a_1(b) \to 1$ as $b \to \infty$, $a_2(b)/b \to 1$ as $b \to \infty$ so that $\tilde{v}_b > 1$ on essentially the whole ball $B(b)$. If v_b is extended to be zero outside $B(b)$ one calculates from (2.12) and (2.13) that $\|v_H - v_b\| \to 0$ as $b \to \infty$ and so we shall be interested in solutions emanating from $(0,v_b)$. The explicit estimates

$$\left.\begin{aligned} \|v_b\|^2 &\simeq \frac{40}{7} \\ \|\tilde{v}_b\|^2 &\simeq \frac{12}{7} b^7 \end{aligned}\right\} \text{as } b \to \infty \tag{2.14}$$

can be found in Appendix B of [3].

The regularity estimates given above ensure that the map $(k,v) \to N(k,v;b)$ is compact from $[0,\infty) \times E_c(b)$ into $E_c(b)$. Unfortunately N is not continuous since convergence in $E_c(b)$ does not satisfactorily control the level sets on which $v(r,z) = 1 + k/r^2$. Hence, a degree argument for the equation $v - N(k,v;b) = 0$ is not immediately applicable. This difficulty can be surmounted by smoothing out the discontinuity in f_0.

2.3. The regularized problem for finite b.

For each $\delta > 0$ let $f_\delta(t)$ be the piecewise linear function

$$f_\delta(t) = \begin{cases} 0 & , \quad t < 0 , \\ t/\delta & , \quad 0 \le t \le \delta , \\ 1 & , \quad t > \delta . \end{cases}$$

For each $(k,v) \in [0,\infty) \times E_c(b)$ and for $b > 0$, $\delta > 0$, let $w = N(k;v;b,\delta) \in E_c(b)$ denote the unique solution of

$$\langle u,w\rangle_E = \frac{15}{2\pi^2} \int_{B(b)} f_\delta\left(v - 1 - \frac{k}{r^2}\right)u, \quad \forall\, u \in E(b) \tag{2.15}$$

and define $N(k,v,b,0)$ to be $N(k,v,b)$ from (2.10). The added regularity of f_δ makes

the map $N(\cdot,\cdot;b,\delta) : [0,\infty) \times E_c(b) \to E_c(b)$ continuous as well as compact, and a degree-theoretic argument is suitable for the equation

$$v - N(k,v;b,\delta) = 0 \quad . \tag{2.16}$$

When $k = 0$, equation (2.16) is equivalent to

$$\begin{aligned} -\Delta v &= 15\, f_\delta(v-1) \quad \text{in} \quad B(b) \\ v &= 0 \quad \text{on} \quad \partial B(b) \quad . \end{aligned} \tag{2.17}$$

Since f_δ is Lipschitz continuous any solution lies in $C^{2+\alpha}(\overline{B(b)})$ for each $\alpha \in (0,1)$. Since $f_\delta \geq 0$, a solution v is non-negative and, in fact, must be positive in $B(b)$ with a maximum larger than unity or be identically zero, by the strong maximum principle. From [13], v is a function of $\rho = \left(\sum_1^5 x_i^2\right)^{1/2}$ and $v'(\rho) < 0$ for $\rho \in (0,b]$. It follows that v satisfies the ordinary differential equation

$$\begin{aligned} -\frac{1}{\rho^4}\frac{d}{d\rho}\left(\rho^4 \frac{dv}{d\rho}\right) &= 15\, f_\delta(v(\rho) - 1),\ \rho \in (0,b] , \\ v(b) &= 0 \quad . \end{aligned} \tag{2.18}$$

We now prove that if b is sufficiently large and δ is sufficiently small, then (2.18) has a unique solution $v_{b,\delta}$ in a neighborhood of the function v_b given in (2.12).

Lemma 2.2. <u>There exist positive numbers</u> δ_0, b_0, <u>and</u> ϵ_0 <u>such that for each</u> $b > b_0$ <u>and</u> $\delta \in (0,\delta_0]$ <u>the problem</u>

$$\begin{aligned} -\Delta v &= -15\, f_\delta(v-1) \quad \underline{\text{in}} \quad B(b) \quad , \\ v &= 0 \quad \underline{\text{on}} \quad \partial B(b) \quad , \\ \|v-v_b\|_{E_c(b)} &< \epsilon_0 \end{aligned} \tag{2.19}$$

<u>has a unique solution</u> $v_{b,\delta}$.

<u>Moreover,</u>

$$\lim_{\delta\to 0} \|v_{b,\delta} - v_b\|_{E_c(b)} = 0 \quad .$$

Proof. The solution with $\delta > 0$ can be expected to be close to v_b, given in (2.12), which satisfies

$$b^4 v_b'(b) = \frac{-3a^3}{1-c} = -3$$

for b large. For a fixed, large value of b let $v = v(\rho;\sigma,\delta)$ be that solution of (2.18) which vanishes at $\rho = b$ and satisfies

$$b^4 v'(b) = \sigma \quad .$$

The idea is to find a value of σ near -3 for which $v'(0) = 0$, eventually using the implicit function theorem. As long as $v < 1$, $v'(\rho) = \sigma/\rho^4$ and thus

$$v(\rho) = -\frac{\sigma}{3}\left(\frac{1}{\rho^3} - \frac{1}{b^3}\right) \quad \text{on} \quad [\alpha,b]$$

where $\alpha = \alpha(\sigma) = (-3\sigma^{-1} + b^{-3})^{-1/3}$. For σ near -3, $v'(\alpha) < 0$ and so v, assumed to lie in C^1, will be larger than unity on an interval to the left of α. There v satisfies

$$\begin{aligned} v''(\rho) + \frac{4}{\rho} v'(\rho) &= \frac{-15}{\delta}(v(\rho)-1) \quad , \\ v(\alpha) = 1 \quad , \quad v'(\alpha) &= \frac{\sigma}{\alpha^4} \quad . \end{aligned} \tag{2.20}$$

For σ near -3 and b_0 large, α is approximately 1 and we expect (2.20) to be satisfied on an interval $[\beta,\alpha]$ where $\alpha-\beta = \delta/3$ and $v(\beta) = 1+\delta$. To see that this is the case and to find the dependence of β on σ and δ, let

$$v(\rho) = 1 + \delta\, w(s)$$

where $\rho = \alpha-\delta s$. Then $w = w(s;\sigma,\delta)$ satisfies

$$\begin{aligned} \frac{d^2w}{ds^2} - \frac{4\delta}{\alpha-\delta s}\frac{dw}{ds} + 15\,\delta w &= 0 \quad , \\ w(0) = 0 \quad , \quad \frac{dw}{ds}(0) &= -\frac{\sigma}{\alpha^4} \quad . \end{aligned} \tag{2.21}$$

The solution w is analytic in all its variables for $s < \alpha/\delta$, and for $\delta = 0$ is the linear function $w(s) = -\frac{\sigma}{\alpha^4} s$. The value $s = -\alpha^4/\sigma$ gives $w = 1$ and since $\partial w/\partial s \neq 0$

for this value of s, the implicit function theorem yields a unique analytic function $s(\sigma,\delta)$ defined for (σ,δ) in a neighborhood Q of $(-3,0)$ such that

$$w(s(\sigma,\delta);\sigma,\delta) \equiv 1 \quad . \tag{2.22}$$

In this case Q will be independent of b for all b larger than some b_0. Taking Q smaller, if necessary, we can assume $\partial w/\partial s > 0$ for $(\sigma,\delta) \in Q$ and $0 \le s \le s(\sigma,\delta)$. This yields a function v which is monotone decreasing in ρ for $\rho \in [\beta,\alpha]$ where $\beta = \alpha(\sigma) - \delta s(\sigma,\delta)$. By construction, $v(\beta) = 1+\delta$.

Since v is required to be C^1 and $v'(\beta) < 0$, it follows that $v > 1+\delta$ on an interval to the left of β and there satisfies

$$\frac{1}{\rho^4}\frac{d}{d\rho}\left(\rho^4 \frac{dv}{d\rho}\right) = -15 \quad ,$$

yielding $\frac{dv}{d\rho} = -3\rho + \text{const.}/\rho$. We want a slope of zero at $\rho = 0$ and thus

$$\frac{dv}{d\rho} = -3\rho \quad \text{on} \quad [0,\beta] \quad .$$

The condition that derivatives match at $\rho = \beta$, expressed in terms of w, is

$$F(\sigma,\delta) \equiv \frac{dw}{ds}(s(\sigma,\delta);\sigma,\delta) - 3(\alpha(\sigma) - \delta s(\sigma,\delta)) = 0 \quad . \tag{2.23}$$

For $\delta = 0$,

$$F(\sigma,0) = \frac{-\sigma}{\alpha^4(\sigma)} - 3\alpha(\sigma)$$

which vanishes for the choice $\alpha = a$, $\sigma = -3a^5$ corresponding to the solution v_b in (2.12). For later use we note that for this choice of parameters

$$s(\sigma,0) = \frac{-a^4}{\sigma} = \frac{1}{3a} \quad . \tag{2.24}$$

Since $\alpha'(\sigma) = -\alpha^4/\sigma^2$ in general,

$$\begin{aligned}\frac{\partial F}{\partial \sigma}(\sigma,0) &= -\frac{1}{\alpha^4} + \frac{4\sigma}{\alpha^5}\alpha' - 3\alpha' \\ &= -\frac{1}{\alpha^4} - \frac{4}{\alpha\sigma} + \frac{3\alpha^4}{\sigma^2} \quad .\end{aligned}$$

For $\sigma = -3$ and b large, α is approximately 1 so $\partial F/\partial\sigma(-3,0)$ is approximately $\frac{2}{3}$. The implicit function theorem applied to (2.23) yields an analytic function $\sigma(\delta)$, defined for δ in a neighborhood of zero which is independent of b, such that $F(\sigma(\delta),\delta) \equiv 0$. The function

$$v_{b,\delta}(\rho) = v(\rho;\sigma(\delta),\delta)$$

satisfies the equation and boundary condition in (2.19). As regards the distance from $v_{b,\delta}$ to v_b it is clear from the construction that the distance in C^1 approaches zero as $\delta \to 0$. Since $1/\rho^3$ and its gradient are in L^2 at infinity in R^5, the convergence in $E_c(b)$ follows as well, uniformly for all large b. q.e.d.

2.3. The index of the solution $v_{b,\delta}$

Throughout this section it will be assumed that $b > b_0$ and $\delta \in (0,\delta_0]$ as in Lemma 2.2. Then $v_{b,\delta}$ is the only solution of $v - N(0,v,b,\delta) = 0$ in $\overline{J}(b)$, where $J(b)$ is the open ball of radius ε_0 centered at v_b. As all computations in this section are done for $k = 0$ we suppress it writing $N(v;b,\delta)$ or merely $N(v)$ when the emphasis is on the behavior with respect to v. For $\delta > 0$ the Frechet derivative of N at $v_{b,\delta}$ is

$$N'(v_{b,\delta})\gamma = -\Delta^{-1}\cdot 15f'_\delta(v_{b,\delta} - 1)\gamma \tag{2.26}$$

where $f'_\delta(v_{b,\delta} - 1) = \delta^{-1}$ when $v_{b,\delta}(\rho) \in (1,1+\delta)$ and is zero elsewhere. Note that $f_\delta(t)$ itself fails to have a derivative at $t = 0$ and $t = \delta$. However, since $v_{b,\delta}$ has a nonzero gradient where $v_{b,\delta} = 1$ and $1 + \delta$, one can, by viewing N as a map $15f_\delta(v-1)$ from $E_c(b)$ to $L^2(B(b))$ followed by $-\Delta^{-1}$ from $L^2(B(b))$ to $E_c(b)$, verify that (2.26) is the derivative.

Standard theory [16] ensures that the Leray-Schauder degree of $I - N(\cdot)$ relative to zero and $J(b)$ is equal to the degree of $I - N'(v_{b,\delta})$ relative to zero and $E_c(b)$. We will show that 1 is not an eigenvalue of N' so this latter degree is well defined. In fact the constancy of degree follows from the homotopy

$$t \to \gamma - \frac{1}{t}\left[N(v_{b,\delta} + t\gamma) - N(v_{b,\delta})\right],\ t \in [0,1]$$

where, for $t = 0$, the last expression is understood to be $\gamma - N'(v_{b,\delta})\gamma$. Using δ as a

homotopy parameter one sees that the degree of $I - N(\cdot)$ relative to zero and $J(b)$ is the same for all $\delta \in (0,\delta_0]$ and so the degree of $I - N'(v_{b,\delta})$ is constant for all sufficiently small δ and large b. By standard theory the degree $d(b,\delta)$ in question is then $(-1)^m$ where $m = m(b,\delta)$ is the total algebraic multiplicity of eigenvalues of $N'(v_b,b,\delta)$ on the interval $(1,\infty)$. Since N' is selfadjoint m is the total geometric multiplicity associated with the interval.

Theorem 2.3. <u>There exist</u> $b_1 > b_0$ <u>and a positive</u> $\delta_1 < \delta_0$ <u>such that</u> $d(b,\delta) = -1$ <u>if</u> $\delta \in (0,\delta_1]$ <u>and</u> $b > b_1$.

<u>Proof</u>. It will be shown that for all small δ and large b the largest eigenvalue of N' is near $5/3$ while the second largest eigenvalue, counting multiplicity, is bounded above by approximately $5/7$. This will show $m(b,\delta) = 1$ for these parameter ranges. Let $\lambda(b,\delta)$ denote the largest eigenvalue, as before, and let $\mu(b,\delta) < \lambda(b,\delta)$ denote the second largest. Let $u(b,\delta)$ and $v(b,\delta)$ denote corresponding eigenfunctions. It can be assumed that $\langle u,v\rangle_E = 0$.

We begin with a discussion of λ and u though much of it applies to μ and v as well. Recall that $f_\delta' = \delta^{-1}$ precisely on the interval $I = (\beta(b,\delta), \alpha(b,\delta))$ and is zero elsewhere. Let χ denote the characteristic function of I. Then u satisfies

$$\begin{aligned} -\Delta u &= \frac{15}{\lambda\delta}\chi(\rho)u \quad \text{in} \quad B(b) \quad , \\ u &= 0 \quad \text{on} \quad \partial B(b) \quad . \end{aligned} \tag{2.27}$$

Alternatively, for any test function ϕ

$$\langle u,\phi\rangle = \frac{15}{2\pi^2\lambda\delta}\int_\beta^\alpha \rho^4 d\rho \int_{S(\rho)} u(\rho,\Omega)\phi(\rho,\Omega)d\Omega \tag{2.28}$$

where $S(\rho) \subset \mathbb{R}^5$ denotes the sphere of radius ρ and $d\Omega$ is the area element on the unit sphere.

The eigenfunction u will be of one sign, will be radial by symmetrization, and will be harmonic where $\chi(\rho) = 0$. The eigenvalue itself can be characterized by

$$\lambda(b,\delta) = \max_{w \in E_c(b)} \frac{\langle N'(v_{b,\delta})w,w\rangle}{\langle w,w\rangle} = \max \frac{\frac{15}{\delta}\int_\beta^\alpha \rho^4 d\rho \int_{S(\rho)} w^2(\rho,\Omega)d\Omega}{\int_{B(b)}|\nabla w|^2} . \tag{2.29}$$

Recall that as $\delta \to 0$, α and β approach $a = a(b) = 1$ and, from (2.24)

$$\lim_{\delta\to 0} \delta^{-1} \int_\beta^\alpha \rho^4 d\rho = a^4 s(\sigma,0) = \frac{a^3}{3} .$$

From this discussion one would expect that

$$\tilde{u}(\rho) = \begin{cases} \frac{1}{a^3} - \frac{1}{b^3} , & \rho \in [0,a] , \\ \frac{1}{\rho^3} - \frac{1}{b^3} , & \rho \in [a,b] \end{cases} \tag{2.30}$$

is a reasonable trial function for (2.29). With this function as w the quotient in (2.29) is well behaved as $\delta \to 0$ and $b \to \infty$. In the limit its value is

$$\frac{15 \cdot \frac{1}{3} \cdot 1 \quad |\Omega|}{\int_1^\infty \rho^4 |\frac{d}{d\rho}(\frac{1}{\rho^3})|^2 d\rho |\Omega|} = \frac{5}{3}$$

where $|\Omega|$ is the measure of $S(1)$. This suffices to show that λ is bounded below by approximately 5/3 for δ small and b large. In fact one can make sense of the eigenvalue problem in the limit as $\delta \to 0$ and u in (2.30) is the eigenfunction corresponding to the largest eigenvalue. To show this and to pave the way for estimating $\mu(b,\delta)$ we carry out such a limit.

From embedding theory ([17], p. 316)

$$|\int_{S(\rho)} u(\rho,\Omega)\phi(\rho,\Omega)d\Omega \le \|u\|_{L^{8/3}(S(\rho))} \cdot \|\phi\|_{L^{8/5}(S(\rho))}$$

$$\le \text{const } \|u\| \cdot \|\phi\|_{W^{1,10/7}(B(b))} .$$

Hence from (2.28)

$$|\langle u,\phi\rangle| \le \text{const } \|u\| \cdot \|\phi\|_{W^{1,10/7}(B(b))}$$

where the constant is independent of δ. By duality,

$$\|u\|_{W^{1,10/3}(B(b))} \leq \text{const } \|u\| \tag{2.31}$$

for a constant independent of δ. Now pick a sequence $\delta_n \to 0$, $n = 1,2,\ldots$ so that $\lambda(b,\delta_n) \to \lambda(b) = \limsup_{\delta \to 0} \lambda(b,\delta)$. Let u_n denote the corresponding normalized eigenfunction. Pick an $\varepsilon > 0$ and let

$$K_\varepsilon = \overline{B(b)} \cap \{\rho \notin (a - \frac{\varepsilon}{2}, a + \frac{\varepsilon}{2})\} \quad .$$

The functions u_n are harmonic in K_ε for large n and so a subsequence, still denoted u_n, will converge in C^1 on each connected component of K_ε to a harmonic function. On the other hand

$$\int_{a-\varepsilon<\rho<a+\varepsilon} |\nabla u_n|^2 \leq \text{const } \varepsilon^{2/5} \|u_n\|^2_{W^{1,10/3}(B(b))}$$

which, from (2.31), is of order $\varepsilon^{2/5}$, independently of n. It follows that u_n converges in $E_c(b)$ to a nonnegative radial function u which is harmonic in the complement of $S(a)$; i.e., to a multiple of $\tilde{u}(\rho)$ in (2.30). As noted, the traces on spheres are well-behaved and the limiting function satisfies

$$\lambda\langle u,\phi\rangle = \frac{5a^3}{2\pi^2} \int_{S(a)} u(a,\Omega)\phi(a,\Omega)d\Omega \tag{2.32}$$

for all $\phi \in E_c(b)$ where $\lambda = \lambda(b)$. In fact, the limits as $\delta \to 0$ of $u(b,\delta)$ and $\lambda(b,\delta)$ exist, for any subsequence of eigenfunctions will converge to a multiple of $\tilde{u}$ in (2.30).

The discussion of limits holds equally well for $\mu(b,\delta)$ and $v(b,\delta)$ as $\delta \to 0$ yielding a pair (μ,v) satisfying

$$\mu\langle v,\phi\rangle = \frac{5a^3}{2\pi^2} \int_{S(a)} v(a,\Omega)\phi(a,\Omega) \tag{2.33}$$

for all $\phi \in E_c(b)$. Naturally, $\langle u,v\rangle_E = 0$ for the limiting functions. Since all eigenfunctions are harmonic in the complement of $S(a)$ and hence determined by their values on the sphere it is natural to examine more closely their behavior on spheres in $\mathbb{R}^5$. Recall that the functions under consideration are functions of r and z. Consider instead coordinates ρ and s where $\rho = \sqrt{r^2 + z^2}$, as before, and $s = \sin\theta$ where

$r = \rho \cos\theta$ and $z = \rho \sin\theta$. Let U be defined by

$$u(r,z) = u(\rho\sqrt{1-s^2}, \rho s) = U(\rho,s)$$

and let V, Φ correspond to v, ϕ, respectively. We have

$$\langle u,\phi\rangle_E = \langle U,\Phi\rangle_G \tag{2.34}$$

where

$$\langle U,\Phi\rangle_G = \int_{-1}^{1}(1-s^2)ds \int_0^b [\rho^4 U_\rho \Phi_\rho + \rho^2(1-s^2)U_s\Phi_s]d\rho \tag{2.35}$$

and hence the eigenvalue equation (2.32) is expressed as

$$\lambda\langle U,\Phi\rangle_G = 5a^3 \int_{-1}^{1}(1-s^2)U(a,s)\Phi(a,s)ds \tag{2.36}$$

for all Φ for which $\langle\Phi,\Phi\rangle_G < \infty$, with an analogous equation for μ and V. Of course,

$$\langle U,V\rangle_G = 0 \quad . \tag{2.37}$$

The function U corresponding to u is a function of ρ alone and so is constant on $S(a)$ yielding

$$\int_{-1}^{1}(1-s^2)V(a,s)ds = 0 \tag{2.38}$$

from (2.36), (2.37). In fact $V(\rho,s)$ is orthogonal to 1 (with the weight $1-s^2$) on each sphere. To see this let

$$\overline{V}(\rho) = \int_{-1}^{1}(1-s^2)V(\rho,s)ds \quad . \tag{2.39}$$

If in the eigenvalue equation for μ, V one admits only radial test functions $\Phi(\rho)$, the result is

$$\begin{aligned} \mu \int_0^b \rho^4\, \overline{V}_\rho \Phi_\rho &= 5a^3\, \overline{V}(a)\Phi(a) \\ &= 0 \end{aligned}$$

for all such Φ. It follows that $\overline{V}(\rho)$ is a radial, harmonic function on $B(b)$, vanishing for $\rho = a$ and thus identically zero. That is,

$$\int_{-1}^{1}(1-s^2)V(\rho,s)ds = 0 \quad \text{for} \quad \rho \in (0,b) \quad . \tag{2.40}$$

A collection of polynomials which is orthogonal with respect to the weight $1 - s^2$ and complete in the weighted L^2 space on $[-1,1]$ is

$$S_n(s) = \frac{d}{ds} P_n(s) \quad , \quad n = 1,2\ldots$$

where P_n is the n^{th} Legendre polynomial. From the Rodrigues' formula one easily sees that

$$\int_{-1}^{1}(1-s^2)S_nS_m = \int_{-1}^{1}(1-s^2)S_n'S_m' = 0 \quad \text{if} \quad m \neq n$$

while standard formulae ([23], Chap. XV) show that

$$\int_{-1}^{1} (1-s^2)S_n^2 = \frac{2n(n+1)}{2n+1}$$

and

$$\int_{-1}^{1}(1-s^2)(S_n')^2 = \frac{2n(n+1)(n^2+n-2)}{2n+1} \quad .$$

Set

$$V(\rho,s) = \sum_{n=1}^{\infty} c_n(\rho)S_n(s) \tag{2.41}$$

where

$$c_n(\rho) = \frac{2n+1}{2n(n+1)} \int_{-1}^{1}(1-s^2)V(\rho,s)S_n(s)ds \quad .$$

Now $S_1, S_3, S_5,\ldots,$ are even functions of s while $S_2, S_4, S_6,\ldots$ are odd. Since $v(r,z)$ is even in z, $V(\rho,s)$ is even in s and thus $c_2(\rho) = c_4(\rho) =\ldots= 0$. Since $S_1(s) \equiv 1$, (2.40) gives $c_1(\rho) \equiv 0$ and so the sum in (2.41) starts at $n = 3$. The absence of the first two terms in (2.41) makes an effective estimate of μ possible. The eigenvalue equation for μ and V becomes

$$\mu\Big[\sum_{n=3}^{\infty} \frac{2n(n+1)}{2n+1} \int_0^b \rho^4(c_n')^2 + \sum_{n=3}^{\infty} \frac{2n(n+1)(n^2+n-2)}{2n+1} \int_0^b \rho^2c_n^2\Big] = 5a^3 \sum_{n=3}^{\infty} \frac{2n(n+1)}{2n+2} c_n^2(a) \quad . \tag{2.42}$$

Since $n^2 + n - 2 \geq 10$ for $n \geq 3$

$$\sum_{n=3}^{\infty} \frac{n(n+1)}{2n+1} \left\{\mu \int_0^b[\rho^4(c_n')^2 + 10\rho^2c_n^2] - 5a^3c_n^2(a)\right\} \leq 0 \quad . \tag{2.43}$$

A simple variational argument shows

$$c_n^2(a) \leq \Lambda(b)\{\int_0^b[\rho^4(c_n')^2 + 10\rho^2c_n^2]\}$$

for all n where $\Lambda(b) \to 1/7$ as $b \to \infty$. It follows that

$$\mu \leq 5a^3\Lambda(b)$$

and as $b \to \infty$ the upper bound for μ converges to 5/7. Hence for $0 < \delta < \delta_1$ and $b > b_1$, where δ_1 and b_1 are suitable constants, $N'(v_{b,\delta};b,\delta)$ has only the eigenvalue $\lambda(b,\delta) = 5/3$ on the interval $[1,\infty)$. q.e.d.

3. The Existence of Global Branches

3.1. The case $\delta \in (0,\delta_1]$ and $b > b_1$

We return to the equation $\Phi(k,v;b,\delta) \equiv v - N(k,v;b,\delta) = 0$ for $k > 0$. Consider the collection of nontrivial solutions

$$P_{b,\delta} = \{(k,v) \in [0,\infty) \times E_c(b) : \Phi(k,v;b,\delta) = 0 \text{ and } v \not\equiv 0\} \quad .$$

The next result summarizes properties of these solutions.

Theorem 3.1. <u>The set</u> $P_{b,\delta}$ <u>is closed and bounded. Moreover</u> $(k,v) \in P_{b,\delta}$ <u>satisfies:</u>

(a) v <u>is cylindrically symmetric in</u> $\mathbf{R}^5$:

$v = v(r,z)$ <u>where</u> $r^2 = x_1^2 + \cdots + x_4^2$ <u>and</u> $z = x_5$.

(b) $v \in C^{2+\alpha}(\overline{B(b)})$ <u>and</u>

$$|v|_{C^{1+\alpha}(\overline{B(b)})} \le \text{const } \|v\| \le \text{const } b^{7/2} \quad ,$$

$$|v|_{C^{2+\alpha}(\overline{B(b)})} \le \text{const } b^{7/2}/\delta \quad ,$$

$$|k| \le \text{const } b^{11/2} \quad ,$$

<u>where the constants depend on</u> $\alpha \in (0,1)$ <u>but are independent of</u> k, v, b, <u>and</u> δ.

(c) v <u>is an even function of</u> z <u>and</u>

$$\frac{\partial v}{\partial z} < 0 \quad \text{on} \quad \overline{B(b)} \cap \{z > 0\} \quad .$$

<u>Proof</u>. (a) This is just a restatement of $v \in E_c(b)$.

(b) Since $|\Delta v| \le 15$, it follows from [1] and embedding theorems that

$$|v|_{C^{1+\alpha}} \le \text{const}(\|v\| + |v|_{L^{10/3}}) \le \text{const } \|v\|$$

and by an inequality completely analogous to (2.11), $\|v\| \le \text{const } b^{7/2}$. The estimate for the $C^{2+\alpha}$ norm is similar but now depends on the Lipschitz constant $1/\delta$ for f_δ. If $(k,v) \in P_{b,\delta}$ then for some $\tilde{x} \in B(b)$, $v(\tilde{x}) \ge 1 + k/|\tilde{x}|^2 > k/b^2$. Hence $k \le b^2 v(\tilde{x}) \le \text{const } b^{11/2}$.

Note that the boundedness of $P_{b,\delta}$ has now been established. As to its being closed one need merely show that no trivial solution is in the closure. From the arguments above, a limiting solution would have to satisfy $v(\tilde{x}) > 1$ at some point and so is nontrivial.

(c) This follows from [13]. q.e.d.

Let $D_{b,\delta}$ denote the maximal connected subset of $P_{b,\delta}$ containing $(0,v_{b,\delta})$.

Theorem 3.2. Suppose $\delta \in (0,\delta_1]$ and $b > b_1$. Then $D_{b,\delta}$ contains a solution $(0,\tilde{v}_{b,\delta})$ with $\tilde{v}_{b,\delta} \neq v_{b,\delta}$.

Proof. This is a variant of a result of Leray and Schauder [18] and can be shown using the techniques in the paper of Rabinowitz (cf. [22], Lemma 1.2). If the bounded set $D_{b,\delta}$ contains only $(0,v_{b,\delta})$ in the "slice" at $k = 0$, then by the use of a suitable open neighborhood of $D_{b,\delta}$ one can derive a contradiction. For the degree of $\Phi(0,\cdot;b,\delta)$ at $v_{b,\delta}$ (and hence on any large ball in $E_c(b)$) would be -1 by Theorem 2.3 while for k sufficiently large $\Phi(k,\cdot;b,\delta) = 0$ has no solutions and thus has degree zero on every open set.

3.2. The case $\delta = 0$ and $b > b_1$

We fix $b > b_1$ and consider the limit of the branches $D_{b,\delta}$ as $\delta \to 0$. Some definitions are needed. If X is a metric space and $\{A_n\}_{n=1}^{\infty}$ a sequence of subsets of X, then $\liminf A_n$ is defined to consist of points $p \in X$ such that every neighborhood of p has nonempty intersection with all but a finite number of the A_n. In constrast, $\limsup A_n$ consists of points p such that every neighborhood of p has nonempty intersection with infinitely many of the A_n. The following result from Whyburn [24] is useful for taking limits of connected sets.

Lemma 3.3. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of connected sets in a metric space such that

(a) $\bigcup_{n=1}^{\infty} A_n$ is precompact

and

(b) $\liminf A_n \neq \phi$.

Then $\limsup A_n$ *is a compact, connected set.*

Theorem 3.4. *There exists a compact, connected set* $D_b \subset [0,\infty) \times E_c(b)$ *of solutions of* $v - N(k,v,b,0) = 0$. *Moreover*

(a) $D_b \cap (\{0\} \times E_c(b)) = \{(0,v_b)\} \cup \{(0,\tilde{v}_b)\}$ *where* v_b, $\tilde{v}_b$ *are, respectively, the "small" and "large" solutions from section* 2.2 (cf. eq. 2.12).

(b) *If* $(k,v) \in D_b$ *then* v *is cylindrically symmetric in* $\mathbb{R}^5$ *and*

$$|v|_{C^{1+\alpha}(\overline{B(b)})} \leq \text{const } \|v\| \leq \text{const } b^{7/2} ,$$

$$k \leq \text{const } b^{11/2} ,$$

$$\frac{\partial v}{\partial z} < 0 \quad \text{on} \quad \overline{B(b)} \cap \{z>0\} ,$$

where the constants are independent of b, k, *and* v.

Proof. For fixed $b > b_1$ let $\delta_n \in (0,\delta]$ $n = 1,2,\ldots$ be a sequence converging to zero. According to Theorem 3.1 there is a closed bounded set $X \subset \mathbb{R} \times E_c(b)$ which contains D_{b,δ_n} for all n. To use Lemma 3.3 let $A_n = D_{b,\delta_n}$. The bounds on v in $C^{1+\alpha}$ and on k from Theorem 3.1(b) are independent of δ and so the use of Arzela's Theorem shows $\bigcup_{n=1}^{\infty} A_n$ is precompact in X. According to Lemma 2.2, $v_{b,\delta} \to v_b$ as $\delta \to 0$ and so $(0,v_b) \in \liminf A_n$. According to the previous lemma $D_b \equiv \limsup A_n$ is a compact, connected set and contains $(0,v_b)$. With the exception of the strict negativity of $\partial v/\partial z$ the remainder of (b) follows immediately from Theorem 3.1 since the relevant estimates are independent of δ.

To complete the proof of (b) consider an element $(k,v) \in D_b$. It can be assumed that (k,v) is the limit in $\mathbb{R} \times C^1(B(b))$ of a sequence $(k_n,v_n) \in A_n$. Since v_n is bounded in $W^{2,2}(B(b))$ uniformly in δ, by elliptic theory, v is also the weak limit in $W^{2,2}$ of v_n. To show $\partial v/\partial z$ has one sign on $B^+ = B(b) \cap \{z>0\}$ we use a weak form of the maximum principle. Let ϕ be an element of $C_0^\infty(B^+)$ with $\phi \geq 0$. Then

$$\int_{B^+} \nabla \frac{\partial v_n}{\partial z} \nabla\phi = \int_{B^+} (\Delta v_n) \frac{\partial\phi}{\partial z}$$

$$= -15 \int_{B^+} f_{\delta_n}\left(v_n - 1 - \frac{k_n}{r^2}\right) \frac{\partial\phi}{\partial z}$$

$$= 15 \int_{B^+} f'_{\delta_n}\left(v_n - 1 - \frac{k_n}{r^2}\right) \frac{\partial v_n}{\partial z} \phi$$

$$\leq 0$$

since $\partial v_n/\partial z < 0$ in B^+. Taking a limit yields

$$\int_{B^+} \nabla \frac{\partial v}{\partial z} \nabla\phi \leq 0 \quad . \tag{3.1}$$

Since, in the limit $\partial v/\partial z \leq 0$ in B^+, Theorem 8.19 of [14] applied to (3.1) ensures that either $\partial v/\partial z < 0$ in B^+ or $\partial v/\partial z \equiv 0$ there. The latter would imply $v \equiv 0$ in $B(b)$, an impossibility, since each v_n, and hence v, must exceed unity somewhere on $B(b)$. To show that $\partial v/\partial z < 0$ at a point $q \in \partial B(b) \cap \{z>0\}$ note that $v_n(q) = 0$ so $|v_n| \leq \frac{1}{2}$ in a neighborhood Q of q, uniformly in n. Since $f_\delta(v_n - 1 - k/r^2) = 0$ on Q, v_n is harmonic there, as is the limiting function v. Hence $\partial v/\partial z < 0$ at q by the maximum principle.

Next we show that $(k,v) \in D_b$ is a solution of the *limiting equation, that is, that*

$$\int_{B(b)} \nabla v \cdot \nabla\phi = 15 \int_{B(b)} f_0(v - 1 - k/r^2)\phi \tag{3.2}$$

for all $\phi \in E_c(b)$. *We consider the sequence* (k_n, v_n) from the previous paragraph and need merely show convergence of the right-hand member of (3.2) evaluated at (k_n, v_n). Pick an $\varepsilon > 0$ and let

$$T_\varepsilon = \{(r,z) \in B(b) : r > \varepsilon \text{ and } |z| > \varepsilon\} \quad .$$

On T_ε the function $\tilde{v}_n = v_n - 1 - k_n/r^2$ converges in C^1 to $\tilde{v} = v - 1 - k/r^2$ as $n \to \infty$. Since $\partial v/\partial z$ is bounded away from zero on T_ε, so is $\partial v_n/\partial z$ for all large n. Say both are less than $-s < 0$ where $z > 0$. Let ϕ be a smooth function in $E_c(b)$. Pick $\sigma > 0$ and assume n is large enough so that $0 < \delta_n < \sigma$ and $|\tilde{v}_n - \tilde{v}| < \sigma$ on T_ε. Then

$$\int_{T_\varepsilon} f_{\delta_n}(\tilde{v}_n)\phi = \int_{T_\varepsilon \cap \{0<\tilde{v}_n<\sigma\}} f_{\delta_n}(\tilde{v}_n)\phi + \int_{T_\varepsilon \cap \{\tilde{v}_n>\sigma\}} 1 \cdot \phi$$

$$= O(\frac{\sigma}{s}) + \int_{T_\varepsilon \cap \{\tilde{v}>0\}} 1 \cdot \phi + O(\frac{\sigma}{s}) \quad .$$

Since σ is arbitrary,

$$\lim_{n\to\infty} \int_{T_\varepsilon} f_{\delta_n}(\tilde{v}_n)\phi = \int_{T_\varepsilon} f_0(\tilde{v})\phi \quad .$$

But since ε is arbitrary and $\text{meas}(B(b)\setminus T_\varepsilon) = O(\varepsilon)$

$$\lim_{n\to\infty} \int_{B(b)} f_{\delta_n}(\tilde{v}_n)\phi = \int_{B(b)} f_0(\tilde{v})\phi \quad .$$

As smooth functions are dense in $E_c(b)$ the equation (3.2) holds.

For part (a) recall that, by Theorem 3.2 and Lemma 2.2 $D_{b,\delta}$ contains a solution $(0,\tilde{v}_{b,\delta})$ such that $\|\tilde{v}_{b,\delta} - v_b\| > \varepsilon_0 > 0$ with ε_0 independent of δ. A subsequence of $(0,\tilde{v}_{b,\delta_n})$ must converge in $E_c(b)$ to a solution $(0,v)$ with $\|v-v_b\| > \varepsilon_0$. However, $\tilde{v}_b$ from section 2.2 is the only solution for $k = 0$, other than v_b. Hence $(0,\tilde{v}_b)$ must belong to D_b. In fact every subsequence, and hence the whole sequence converges to $(0,\check{v}_b)$. q.e.d.

3.3. The case $\delta = 0$ and $b \to \infty$

Once again we shall use Lemma 3.3 to take limits of solution branches as $b \to \infty$. The main result is

Theorem 3.5. <u>There exists an unbounded, closed, connected set</u> $D \subset [0,\infty) \times E_c$ <u>of nontrivial solutions of</u> $v - N(k,v;\infty,0) = 0$:

$$\int_{R^5} \nabla u \cdot \nabla\phi = 15 \int_{R^5} f_0(v - 1 - \frac{k}{r^2})\phi = 15 \int_{A(k,v)} v, \quad \forall \phi \in E \tag{3.3}$$

<u>where</u> $A(k,v) = \{x \in R^5 : v(x) > 1 + k/r^2\}$. <u>Moreover</u>

(a) $D \cap (\{0\} \times E_c) = \{(0,v_H)\}$ <u>where</u> v_H <u>is given in</u> (2.5).

<u>The following properties hold for any</u> $(k,v) \in D$.

(b) v is cylindrically symmetric, $v \in C^2(\mathbb{R}^5 - \partial A(k,v)) \cap C^{1+\alpha}(\mathbb{R}^5)$ for each $\alpha \in (0,1)$, and

$$|v|_{C^{1+\alpha}} \leq \text{const } \|v\|$$

with a constant depending on α.

(c) The set $A(k,v)$ is bounded while for $|x| \to \infty$, $|v(x)| = O(|x|^{-3})$ and $|\nabla v(x)| = O(|x|^{-4})$.

(d) v is an even function of z and $\partial v/\partial z < 0$ for $z > 0$.

Proof. Each element of $E_c(b)$ is extended to be zero outside $B(b)$ and considered as an element of E_c. Let

$$X_j = \{(k,v) \in [0,\infty) \times E_c : k^2 + \|v\|^2 < j^2\}$$

where j is a fixed integer larger than $\|v_H\|$. Choose a sequence $b_n \to \infty$, $n = 1,2\ldots$ and let $A_n = D_{b_n} \cap X_j$. To show that $\bigcup_{n=1}^{\infty} A_n$ is precompact in $\mathbb{R} \times E_c$ it suffices, as before, to show that a sequence $(k,v) \in A_n$, $n = 1,2,\ldots$ is precompact. One can suppose $k_n \to k > 0$ and that v_n converges weakly to a element $v \in E_c$. To show strong convergence first note that from Theorem 3.4(b), v_n has a $C^{1+\alpha}$ bound independent of n since $\|v_n\| < j$. Hence $\{v_n\}$ converges in C^1 on any compact subset of $\mathbb{R}^5$. It is shown in Theorem 5E of [11] that there is a ball $B(\beta) \subset \mathbb{R}^5$, independent of n, such that $v_n(r,z) < 1 + k_n/r^2$ for (r,z) outside the ball $B(\beta)$. That is, v_n is harmonic outside $B(\beta)$. Since the norms $\|v_n\|$ are uniformly bounded, it is elementary to show that for any $\varepsilon > 0$ there exists a $\gamma > \beta$ such that

$$\int_{\gamma<\rho<b_n} |\nabla v_n|^2 < \varepsilon$$

for all n. This last inequality combined with the convergence on compact sets of $\mathbb{R}^5$ shows $v_n \to v$ in E_c. Recall from section 2.2 that $v_b \to v_H$ as $b \to \infty$ and thus $(0,v_H) \in \liminf A_n$. Lemma 3.3 yields a compact, connected set D^j which can easily be seen, as before, to satisfy the equation (3.3). The set D^j contains $(0,v_H)$ and must also contain a solution (k,v) satisfying $k^2 + \|v\|^2 = j^2$. For the branch D_{b_n}

connecting v_{b_n} to $\tilde{v}_{b_n}$ must, for all large n, intersect the set where $k^2 + \|v\|^2 = j^2$, implying $\limsup A_n$ contains an element of the same closed set. Since each solution v must exceed unity at some point, it is nontrivial. The unbounded solution set results from defining

$$D = \bigcup_j D^j$$

where the union is over all large integers j.

(a) This follows since v_H is the only cylindrically symmetric solution of (2.46) for $k = 0$ according to [3].

(b) This follows from the previous theorem and standard estimates.

(c) It follows from the earlier discussion that $A(k,v) \subset B(\beta)$. The decay estimates are those for harmonic functions.

(d) One uses the maximum principle, as before. q.e.d.

The mapping $v \to \psi = r^2 v \in H(\Pi)$ yields an unbounded, closed, connected set C of solutions as required for the proof of Theorem 1.1(a).

4. Properties of D and some Conjectures

In this section, we consider briefly the sense in which D is unbounded in $[0,\infty) \times E_c$. Some numerical results of Norbury [21], when suitably rescaled, suggest there exists a function $h : [0,\infty) \to E_c$ such that $D = \{(k,h(k)) : k \in [0,\infty)\}$. When $k = 0$, the function $h(0)$ is Hill's solution v_H, while $h(k)$ for large k corresponds to a family of vortex rings of small core shown to exist by Fraenkel [9], [10]. As $k \to \infty$, the cores are asymptotic to smaller and smaller circles centered on the r-axis, and whose centers become unbounded. According to Norbury's calculations, these vortex rings are simply-connected in Π.

Our techniques do not allow us to confirm the numerical predictions implicit in the work of Norbury that D is unbounded in the k-direction. Nor have we shown that the cores $\{(r,z) \in \Pi : \psi(r,z) > r^2 + k\}$ are always simply-connected. This is only known for Hill's solution and for solutions in D near to it [4]. One might hope to use the connectedness of D to show that a sequence of cores could not lose their simply-connectedness by pinching together.

In order to show that D is unbounded in the k-direction, we would need to establish bounds of the form

$$\sup_{0<k<\tilde{k}} \|v\| < \infty \tag{4.1}$$

for any solution $(k,v) \in D$ and any $\tilde{k} > 0$. If (4.1) fails to hold, then there would exist elements $(k_n,v_n) \in D$ with

$$k_n \to k \in [0,\infty) \quad \text{and} \quad \|v_n\| \to \infty \quad \text{as} \quad n \to \infty \ . \tag{4.2}$$

There are two ways in which $\|v_n\|$ may become unbounded; the functions v_n may develop a singularity on some compact set or they may lose their decay rate at infinity. As regards the first point, we record that

$$\sup_{0<k<\tilde{k}} |v|_{C^{1+\alpha}(\mathbb{R}^5)} \le \text{const} \tag{4.3}$$

for any solution (k,v) and any $\tilde{k} > 0$. Here the constant depends only on $\alpha \in (0,1)$ and $\tilde{k}$. The proof of (4.3) is very technical and will not be presented here. We claim that if

(4.2) occurs, then the core $A(k_n,v_n) = \{(r,z) \in \mathbb{R}^5 : v_n(x) > 1 + k_n/r^2\}$ must become unbounded. Indeed, if not, then the v_n would be harmonic outside some fixed compact set, and would have a uniform decay rate at infinity: $|\nabla v_n| = O((r^2 + z^2)^{-2})$.

For each $k > 0$, our equation $-\Delta v = 15\, f_0(v-1-k/r^2)$ has a singular solution v_s with infinite norm. Here v_s is a function of r alone, and is given by

$$v_s(r) = \begin{cases} 2 \;, & 0 < r < \sqrt{k}\,, \\ 2 + \dfrac{15k}{4} - \dfrac{15r^2}{8} - \dfrac{15}{8}\dfrac{k^2}{r^2}\,, & \sqrt{k} < r < \sqrt{\dfrac{8}{15} + k}\;, \\ \dfrac{8 + 30k}{15r^2}\;, & r > \sqrt{\dfrac{8}{15} + k}\;. \end{cases} \tag{4.4}$$

Note that v_s is the unique z-independent solution of our equation with $v(r) \to 0$ as $r \to \infty$. There are other solutions which are also functions of r alone, but they do not vanish as $r \to \infty$. There are a number of partial results which suggest that if (4.2) occurs, then the v_n converge on compact sets to the singular solution (4.4). However, we do not present them here since it is our firm conviction that (4.2) does not occur. In such a case, equation (4.1) would hold and the branch D would be unbounded in the k-direction.

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A steady vortex ring of prescribed strength and propagation speed can be described in terms of a Stokes stream function Ψ. A flux constant k measures the flow through the center of the axisymmetric vortex ring. For $k = 0$, Hill in 1894 found an explicit solution for the semi-linear elliptic equation satisfied by Ψ. In this paper it is shown that there is an unbounded, closed, connected branch of solutions emanating from Hill's vortex in the space of pairs (k,Ψ).

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